ANALYSIS OF NUMERICAL AND ANALYTICAL SOLUTIONS FOR BOUNDARY VALUE PROBLEMS IN THERMAL ENGINEERING

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Synopsis

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1. Introduction:

Most substances expand when their temperature is raised and contract when cooled, and for a wide range of temperatures this expansion or contraction is proportional to the temperature change. This proportionality is expressed by the coefficient of linear thermal expansion which is defined as the change in length which a bar of unit length undergoes when its temperature is changed by 1°.

If free expansion or contraction of all the fibers of a body is permitted, no stress is caused by the change in temperature. However, when the temperature rise in a homogeneous body is not uniform, different elements of the body tend to expand by different amounts and the requirement that the body remain continuous conflicts with the requirement that each element expand by an amount proportional to the local temperature rise. Thus, the various elements exert upon each other a restraining action resulting in continuous unique displacements at every point. The system of strains produced by this restraining action cancels out all, or part of, the free thermal expansions at every point so as to ensure continuity of displacement. This system of strains must be accompanied by a corresponding system of self-equilibrating stresses. These stresses are known as thermal stresses.

A similar system of stresses may be induced in a structure made of dissimilar materials even when the temperature change throughout the structure is uniform. Also, if the temperature change in a homogeneous body is uniform and external restraints limit the amount of expansion or contraction, the stresses produced in the body are termed thermal stresses.

Transient thermal stresses are an important consideration in production processes involving large temperature changes. Recently, thermal stresses have also become significant in design problems related to microelectronic devices through their effects on material properties and system parameters. To calculate the thermal stresses, three kinds of methods are available. The first is the analytical method, in which the elasticity theory is used to find the exact solution. The second approach consists of some kind of approximate technique, such as a perturbation procedure. The third method is the use of a numerical process, such as a finite-difference or a finite-element method.

1.1 Basic Terminology

1.1 .1 Material Classification

The materials are classified depending upon their transport properties like thermal conductivity, thermal diffusivity etc as isotropic, anisotropic, and orthotropic.

- **Isotropic material:** Solids which have transport properties that do not depend on direction are said to be isotropic.

  Properties:

  - The thermal conductivity remains the same in all the directions.
  - The heat flux is directly proportional to the temperature gradient at that position.
  - The heat flux vector is normal to the isothermal surface passing through that position.

- **Anisotropic material:** Solids that have properties exhibit directional characteristics are said to be anisotropic.

  Properties:

  - The thermal conductivity is not same in all the directions.
  - The heat flux is not simply proportional to the temperature gradient, each of the components the temperature is weighted according to the anisotropy of the particular material.
  - The heat flux vector is not normal to the surface and it is not parallel to the temperature gradient.

- **Orthotropic material:** Solids that have properties completely defined in terms of three perpendicular directions. Those three principal material directions are usually defined by a user defined coordinate system or a user defined reference plane.

  It is a particular case of anisotropic material.

1.1.2 Modes of Heat Transfer

Heat can be transferred in three different modes: Conduction, Convection, and Radiation. All modes of heat transfer require the existence of temperature difference and all the modes are from high temperature medium to a lower temperature one.

- **Conduction:** Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles. Conduction can take place in solids, liquids, or gases.

According to French mathematician Jean Baptiste Joseph Fourier (1768-1830), the heat flux vector is proportional to the temperature gradient, that is,

$$\bar{q}(\bar{r},t) = -k\ \nabla T(\bar{r},t)$$

where, $\bar{q}(\bar{r},t)$ is the heat flux vector and

$\nabla T(\bar{r},t)$ is the temperature gradient vector

the minus sign is introduced in order to make the heat flow a positive quantity,

the proportionality factor k is called the thermal conductivity of the material having unit W/mK.

- **Convection:** Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion. The faster the fluid motion, the grater the convection heat transfers.

  The rate of convection heat transfer is proportional to the temperature difference, and it is expressed by Newton's law of cooling (1642 -1727),

$$\bar{q}.\bar{n} = h.\left(T - T_a\right)$$

  where, $\bar{n}$ is the unit outward normal to the surface and

  T is the surface temperature,

  T_a is the temperature of the fluid sufficiently away from the surface,

  h is the convection heat transfer coefficient in W/m^2K.

- **Radiation:** Radiation is the energy emitted by matter in the form of electromagnetic waves as a result of change in electronic configuration of the atoms or molecules. All the bodies above absolute zero temperature emit thermal radiation.

  The maximum rate of radiation that can be emitted form a surface at thermodynamic temperature T is given by the Stefan Boltzmann law as,

$$\bar{q}_{\max} = \sigma T^4$$

  where, $\sigma = 5.67 \times 10^{-8}$ W/m^2K^4 is the Stefan Boltzmann constant.

1.2. Mathematical Formulation

The Differential Equation of Heat Conduction

The differential equation of heat conduction for stationary, homogeneous and isotropic solid with heat generation within the body is given as,

$$\rho c_p \frac{\partial T\left(\bar{r},t\right)}{\partial t} = k\nabla^2 T\left(\bar{r},t\right) + g\left(\bar{r},t\right)$$

where, ρ is the density of the solid,

c_p is the specific heat and

$g\left(\bar{r},t\right)$ is the heat generation function of both position and time.

k is the thermal conductivity of a material.

Special cases

- Thermal conductivity is uniform that is, independent of position and temperature.

$$\frac{1}{\alpha}\frac{\partial T\left(\bar{r},t\right)}{\partial t} = \nabla^2 T\left(\bar{r},t\right) + \frac{g\left(\bar{r},t\right)}{k}$$

where, α is called thermal diffusivity of the medium and is defined as,

$$\alpha = \frac{k}{\rho c_p}$$

- Uniform thermal conductivity, no heat sources.

$$\frac{1}{\alpha}\frac{\partial T\left(\bar{r},t\right)}{\partial t} = \nabla^2 T\left(\bar{r},t\right)$$

which is called as the Fourier equation of heat conduction or the diffusion equation.

- Uniform thermal conductivity, steady state, heat generation within the solid.

$$\nabla^2 T\left(\bar{r}\right) + \frac{g\left(\bar{r}\right)}{k} = 0$$

which is Poisson's equation.

- Steady state no heat generation.

$$\nabla^2 T\left(\bar{r}\right) = 0$$

which is Laplace equation.

In different coordinate systems, the three-dimensional, homogeneous differential equation of heat conduction with no heat generation within the solid are given as:

- **In Cartesian coordinate system:**

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where, $T \equiv T(x, y, z, t)$

❖ **In Cylindrical coordinate system:**

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where, $T \equiv T(r,\phi,z,t)$

❖ **In Spherical coordinate system:**

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin\psi}\frac{\partial}{\partial \psi}\left(\sin\psi\frac{\partial T}{\partial \psi}\right)-\frac{1}{r^2\sin^2\psi}-\frac{\partial^2 T}{\partial \phi^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where, $T \equiv T(r,\psi,\phi,t)$

1.3. Theory of Elasticity

The first name to be linked with the history of the theory of elasticity is Galileo Galilei (1564-1642), an Italian astronomer, mathematician and physicist. The problem of cantilever beam is associated with Galileo's name.

Deflection of elastic surfaces was investigated by Euler, Jacques Bernoulli (1759-1789), Gustav Robert Kirchhoff (1824-1887).

The fundamental contribution to the classical theory was made by Hooke R., Navier C.L.M.H., Cauchy A.L., Lame G., Green G., Clapeyron B.P.E.In 1821; Navier established the basic equations of elasticity. Cauchy developed the theory of stress and strain.

The coupling between the strain and temperature fields was first studied by Duhamal (1837) who derived equations for the distribution of strain in an elastic medium subjected to temperature gradients.

Neumann (1855) and several others worked on Duhamals theory and solved number of interesting problems. Biot (1956) gives satisfactory derivation of the equation of thermal conductivity. Voigt (1910) and Jeffrey's (1930) made attempts for thermodynamical justification of these equations.

❖ **Stress:** When a body is deformed the internal force called into play per unit area to restore it to its original state is called as stress. Since the deforming body is at rest, the stress must be equal and opposite to the deforming force per unit area and its unit is N/m^2.

❖ **Strain:** The change per unit amount is called strain. Strain is the fractional deformation produced in a body when it is subjected to a set of deforming forces. Strain being ratio has no unit. There are following three types of strains,

- **Longitudinal or Tensile Strain:** It is defined as the change in length per unit original length.
- **Shear or Shearing Strain:** It is defined as the ratio of the relative displacement between two layers, under the action of tangential force to the distance between them, the distance being measured at right angle to the direction of stress.
- **Volumetric Strain:** It is defined as the change in volume per unit original volume, without any change in shape.

❖ **Hooke's Law:**

The relation between applied forces and the extension of a body is created to Robert Hooke (1635-1703). The six relations between components of the stress tensor and strain tensor are referred to as generalized Hook's law.

According to this law strain is directly proportional to the stress within elastic limits. i.e.

$$\text{Stress } \alpha \text{ Strain}$$

$$\text{Stress} = E \text{ Strain}$$

where, E is Modulus of elasticity. The unit of E is N/m^2.

Corresponding to the three types of strain we have three modulus of elasticity

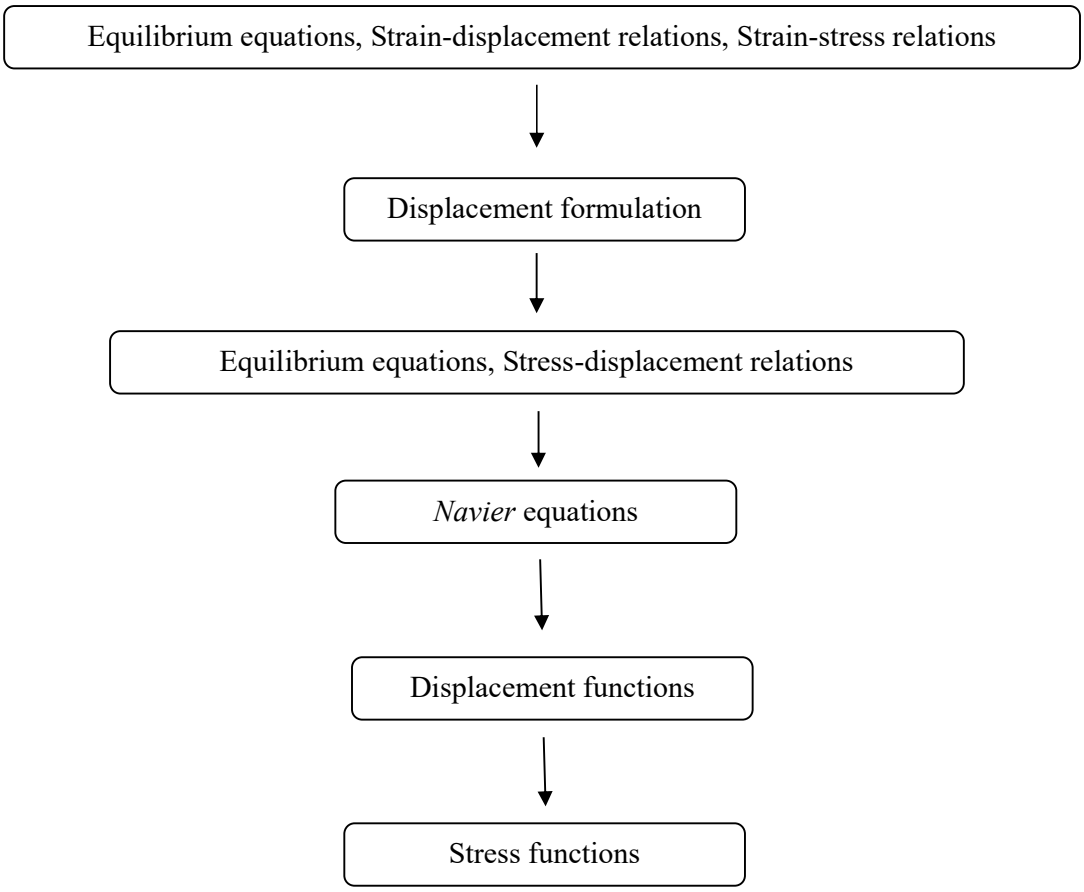
❖ **Young's Modulus:** The ratio of the longitudinal stress to the longitudinal strain within elastic limit is called as Young's Modulus.

❖ **Modulus of rigidity:** The ratio of tangential force per unit area to the angular deformation produced is called Modulus of rigidity.

❖ **Bulk Modulus:** The ratio of stress to volume strain is called Bulk Modulus.

❖ **Development of equations**

The displacement and stress functions are developed according to the following algorithm,



❖ **Classification of Thermoelastic Problems**

Thermoelastic problems are classified as,

- Direct Thermoelastic Problems: Determination of thermal stresses, strain, and deflection, deformation etc due to the temperature and heat flux applied on the boundary surfaces.
- Inverse Thermoelastic Problems: Determination of temperature and heat flux due to one of the thermal effects like thermal stresses, strain, and deflection etc.

4. Thermal Stresses

❖ **Cartesian Coordinates:**

Here the plate is assumed sufficiently thin and considered free from traction. Since the plate is in a plane stress state without bending. Airy stress function method is applicable

to the analytical development of the thermoelastic field. The fundamental equation as in given by the relation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 U=-\alpha E\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T$$

where α, E and U are linear coefficient of the thermal expansion, Young's modulus elasticity of the material of the plate and Airy stress function respectively. The displacement components u_x and u_y in the X and Y direction are represented in the integral form and the stress components in terms of U are given by

$$u_x=\int\left\{\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2}-\nu\frac{\partial^2 U}{\partial x^2}\right)+\alpha T\right\}dx,$$

$$u_y=\int\left\{\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2}-\nu\frac{\partial^2 U}{\partial y^2}\right)+\alpha T\right\}dy,$$

$$\sigma_{xx}=\frac{\partial^2 U}{\partial y^2}$$

$$\sigma_{yy}=\frac{\partial^2 U}{\partial x^2}$$

and

$$\sigma_{xy}=-\frac{\partial^2 U}{\partial xy}$$

where ν is the Poisson's ratio of the material of the plate.

- **Cylindrical Coordinates:**

The differential equation governing the displacement potential function $\phi(r,z)$ is given in as,

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=K\tau$$

where K is the restraint coefficient and temperature change $\tau=T-T_i$, T_i is initial temperature. Displacement function ϕ is known as Goodier's thermoelastic displacement potential.

The displacement function in the cylindrical coordinate system are represented by the Michell's function defined in as

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r dz}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}$$

The Michell's function M must satisfy

$$\nabla^2 \nabla^2 M = 0$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}.\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The components of the stresses are represented by the thermoelastic displacement potential ϕ and Michell's function M as

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right]$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \phi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right]$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right]$$

And $$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right]$$

where G and v are the shear modulus and Poisson's ratio respectively.

1.5. The Boundary Conditions

The differential equation of heat conduction will have numerous solutions unless a set of boundary conditions and initial condition are prescribed. For convenience the linear boundary conditions are separated into three groups:

- **Boundary conditions of first kind**

Temperature is prescribed along the boundary surface and for the general case it is a function of both time and position. Special cases include temperature at the boundary is a function of position only or a function of time only or a constant.

If the temperature at the boundary surface vanishes i.e. T=0, on the boundary surface s_i.

This special case is called as the homogeneous boundary condition of first kind. A boundary surface which is kept at constant temperature T_o, also a boundary condition of first kind.

Moreover called as the Dirichlet boundary condition, named after Johann Peter Gustav Lejeune Dirichlet (1805–1859).

- **Boundary conditions of second kind**

The normal derivative of temperature is prescribed at the boundary surface it may be a function of both time and position. For a boundary surface that fits the coordinate surface of the orthogonal coordinate system it is given in the form,

$$\frac{\partial T}{\partial n_i} = f_i\left(\overline{r_s}, t\right)$$, on the boundary surface s_i.

Special cases include the normal derivative of temperature is a function of position only or a function of time only or a constant. If the normal derivative of temperature at the boundary surface vanishes on the boundary surface s_i . This special case is called as the homogeneous boundary condition of second kind.

Also called as the Neumann boundary condition, named after Carl Neumann

- **Boundary conditions of third kind**
- A linear combination of temperature and normal derivative of temperature is prescribed at the boundary surface. For a boundary surface that fits the coordinate surface of the orthogonal coordinate system is given by,

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i\left(\overline{r_s}, t\right)$$, on the boundary surface s_i.

A special case is $k_i \frac{\partial T}{\partial n_i} + h_i T = 0$ which is called as boundary condition of third kind also called as the mixed boundary condition.

1.6. Mathematical Methods

❖ Integral Transform Method

In solving the transient heat conduction problems, it is useful to apply Laplace transform to remove time variable from the partial differential equation. However, in many problems it is convenient to apply an integral transformation that removes the space variable from the partial differential equation. The integral transform technique is especially attractive for transient and steady state heat conduction problems in that it treats all the space variables in same manner and has no inversion difficulty.

We define the integral transform as,

$$\overline{T}(\beta_m, t) = \int_{x=0}^{L} K(\beta_m, x').T(x', t)dx'$$

and the Inversion formula as,

$$T(x,t) \sum_{m=1}^{\infty} K(\beta_m, x).\overline{T}(\beta_m, x)$$

where., $K(\beta_m, x)$ is the kernel and can be designed according to the boundary conditions of the given problem.

❖ Differential Transform Method

Differential Transform Method (DTM) is based on the Taylor's series expansion. In 1986, the method was first introduced by Zhou to solve linear and nonlinear initial value problems associated with electrical circuit analysis. The DTM obtains an analytical solution in the form of series.

The basic definitions of differential transform method are introduced as follows,

If a function u(x,t) is analytic and continuously differentiable function with respect to time t and space variable x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}$$

Where, $U_k(x)$ is the transformed function.

The differential inverse transform of $U_k(x)$ is defined as follows,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k$$

Then combining the above equations,

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0}$$

❖ **Finite Difference Method**

- **An explicit method**

Using forward difference in the time domain, the time derivative of temperature T(x,t) is given by,

$$\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t} + O(\Delta t)$$

Finite difference expression for partial derivative with respect to the space variable is given as,

and

$$\frac{\partial T}{\partial x} = \frac{T_{i+1}^n - T_i^n}{(\Delta x)} + O(\Delta x)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i-1}^n + T_{i+1}^n - 2T_i^n}{(\Delta x)^2} + O(\Delta x)^2$$

Substituting these values in the one dimensional heat equation, we get the recursive relation as,

$$T_i^{n+1} = rT_{i-1}^n + (1-2r)T_i^n + rT_{i+1}^n \ , i = 1,2,.....N-1, n = 0,1,2,.....$$

where, $r = \frac{\alpha \Delta t}{(\Delta x)^2}$ and truncation error of order $O(\Delta t) + O(\Delta x)^2$

The boundary conditions involving derivatives which should be expressed in finite

The entire set of equations are stable for $0 < r \equiv \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

- **Crank Nicolson implicit method**

The Crank Nicolson finite difference method was invented by John Crank and Phyllis

Nicolson. They originally applied to the heat equation and they approximated the solution of the heat equation on some finite grid by approximating the derivatives in space variable x and time by finite difference.

The one dimensional heat equation is approximated numerically by,

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{1}{2}\alpha \left[\frac{T_{i-1}^{n+1} + T_{i+1}^{n+1} - 2T_i^{n+1}}{(\Delta x)^2} + \frac{T_{i-1}^{n} + T_{i+1}^{n} - 2T_i^{n}}{(\Delta x)^2} \right]$$

The above finite difference equations are rearranged in the following form,

$$-rT_{i-1}^{n+1} + (2+2r)T_i^{n+1} - rT_{i+1}^{n+1} = rT_{i-1}^{n} + (2-2r)T_i^{n} + rT_{i-1}^{n}, i = 0,1,...,N$$

This relation is called as Clank-Nicolson algorithm and it requires a simultaneous solution of all the equations at each time step.

2. Review of Literature

In 1960, Boley et al [1] studied thermally induced beam and plate vibrations. They solved the problem of normal defection of an axisymmetrically heated circular plate in the case of fixed and simply supported edge.

In 1972, Roy Choudhary [2] determined quasi-static thermal stresses in thin circular plate due transient temperature applied along the circumference of a circle over the upper face, also in his second problem he determined thermal deflection in a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper surface, while the lower surface is kept at zero temperature and circular edge is thermally insulated.

In 1979, Leon Y.Bahar et al [3], applied the Laplace transforms with respect to time of the governing equations of one-dimensional thermoelasticity and studied applications to problems pertaining to a half-space and a layer of finite thickness are presented.

In 1989, Noda et al [4] discussed an analytical method for inverse problem of three dimensional transient thermoelasticity in a transversely isotropic solid by integral transform technique with newly designed potential function and illustrated the practical applicability of the method.

In 1991, Chen et al [5] used the finite element method to analyze the thermal buckling of laminates with central circular hole under uniform and non-uniform temperature distribution.

In 1998, Deshmukh et al [6] studied inverse problem on hollow cylinders, circular plates and annular disc by applying integral transform technique and illustrated numerically. Also they considered thermoelastic inverse problem in an anisotropic cylinder and determined the temperature and thermal stresses on the outer curved surface of a cylinder.

In 2002, Hibbeler [7] provided an analysis of the stress and deflection of a laminated plate, which is subjected to an axisymmetric temperature distribution, caused due to heat input along inner boundary. The solution is based on a thermoelastic analysis of the plate.

In 2002, Senthil et al [8] obtained an exact solution for three-dimensional deformations of a simply supported functionally graded rectangular plate subjected to mechanical and thermal loads on its top and/or bottom surfaces. Suitable temperature and displacement functions that identically satisfy boundary conditions at the edges are used to reduce the partial differential equations governing the thermomechanical deformations to a set of coupled ordinary differential equations in the thickness coordinate, which are then solved by employing the power series method.

In 2004, Qian et al [9] studied transient thermoelastic deformations of a thick functionally graded plate with edges held at a uniform temperature and either simply supported or clamped. Either the temperature or the heat flux is prescribed on the top surface of the plate with the bottom surface of the plate kept at either a uniform temperature or thermally insulated. Stresses and deformations induced due to the simultaneous application of the transient thermal and mechanical loads are also computed. The problem is solved by using a higher order shear and normal deformable plate theory and a meshless local Petrov–Galerkin method.

In 2005, Shao [10] used a multi-layered approach based on the theory of laminated composites, the solutions of temperature, displacements, and thermal/mechanical stresses in a functionally graded circular hollow cylinder are presented. The cylinder has finite length and is subjected to axisymmetric thermal and mechanical loads. The material properties are assumed to be temperature-independent and radially dependent, but are assumed to be homogeneous in each layer.

In 2007, Kulkarni et al [11] determined the transient thermal stresses in thick annular disc. A thick annular disc is considered having zero initial temperature and subjected to arbitrary heat flux on the upper and lower surfaces where has the fixed circular edges are at zero

temperature. The governing heat conduction equation has been solved by using integral transform technique.

In 2007, Shi-Rong et al [12] studied numerically the axisymmetric vibrations of a statically buckled polar orthotropic circular plate due to uniform temperature rise. Effects of geometric nonlinearities have been incorporated into the problem formulation. By assuming that the amplitude of plate's vibration and the additional strains induced in it are infinitesimal, and its response harmonic, the non-linear partial differential equations are reduced to two sets of coupled ordinary differential equations; one for the thermal post-buckling, and the other for linear vibrations of the plate superimposed upon the post-buckled configuration. The plate's boundary is taken to be either clamped or simply supported but restrained from moving in the radial direction. The two sets of coupled boundary value problems are solved numerically by a shooting method.

In 2008, Hsin-Ping Chu et al [13] employed the differential transform to discuss the behaviors of nonlinear heat conduction problem. A hybrid method of differential transform and finite difference approach is proposed to solve the transient responses of a nonlinear heat conduction problem. Different parameters of the equation and boundary conditions are considered to verify the feasibility of the proposed method to such problems. Simulation results are illustrated and discussed in comparison with the linear case. The results show that the hybrid method can achieve good results for such problems.

In 2008, Kulkarni et al [14] determined the unknown temperature and thermal stresses on the upper surface of a thick circular plate. A thick circular plate is subjected to arbitrary known interior temperature under steady state. The fixed circular edge and lower surface of the circular plate are thermally insulated. The governing heat conduction equation has been solved by using Hankel transform methods. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

In 2009, Viliani et al [15] studied the active buckling control of smart functionally graded (FG) plates using piezoelectric sensor/actuator patches. A simply supported FG rectangular plate which is bonded with piezoelectric rectangular patches on the top and/or the bottom surface(s) as actuators/sensors is considered. When a constant electric charge is imposed, the governing differential equations of motion are derived using the classical laminated plate

theory (CLPT). The solution for the equation of motion is obtained using a Fourier series method and the effect of feedback gain on the critical buckling load for PZT-4 is studied.

In 2009, Dhawan [16] observed temperature decay in an aluminium plate using Galerkin finite element method for 2D transient heat conduction equation. Taking Δt of 0.001 hr, temperature variation is studied using unconditionally stable first order and second order accurate schemes - backward Euler and modified Crank- Nicholson respectively. A comparative study has been made taking different combinations of meshes and numerical schemes. Desired calculations are made taking into account the properties of aluminium plate.

In 2010, Le-Chung Shiau et al [17] studied the thermal buckling behavior of composite laminated plates by making the use of finite element method. The thermal buckling mode shapes of cross-ply and angle-ply laminates with variousE1/E2 ratios, aspect ratios, fiber angle, stacking sequence and boundary condition were studied in detail.

In 2011, Varghese et al [18] applied integral transformation techniques to study thermoelastic response of a thick annular disc, in general in which sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type.

In 2012. Mohsen Torabi et al [19] in this paper, the nonlinear heat transfer equation is investigated by considering a variable specific heat coefficient. The calculations are carried out by using the differential transformation method (DTM), which is a seminumerical analytical solution technique. Using the DTM, the nonlinear constrained governing equations are reduced to recurrence relations and related initial conditions are transformed into a set of algebraic equations. The principle of differential transformation is briefly introduced, and is then applied to the aforementioned equation.

In 2013, K. Saravanakumar et al [20], studied homotopy analysis method to obtain an approximation of analytical expression of fin efficiency of convective straight fins with temperature-dependent thermal conductivity.

In 2014, Te Wen Tu et al [21], in this paper, an alternative approach is proposed to develop the analytic solution for the one dimensional heat conduction with one mixed type boundary condition and general time-dependent heat transfer coefficient

In 2014, Sobhan Mosayebidorcheh et al [22] in this paper, differential transform method (DTM) is used to solve the nonlinear heat transfer equation of a fin with the power-law temperature-dependent both thermal conductivity and heat transfer coefficient

In 2015 B. B. Pandit et al [23], in this paper deals with the finite difference method for the solution of non-linear boundary value problem. The thermoelastic behavior due to radiation under transient temperature conditions was discussed.

In 2016, Marianito R et al [24], considered a one-dimensional multilayer diffusion problem subject to nonhomogeneous boundary conditions. Unlike previous results that used a separation of variables technique to solve such problems with homogeneous boundary conditions, here we use a Laplace transform approach.

In 2018, The net residual stresses in the films of American Superconductor tape were investigated using a displacement-energy model (DEM) solution approach by Stephen Ogbonna Mbam et al. [25]. The DEM solution method was created to forecast the thermal mismatch and inherent film stresses in coated conductor films.

In 2019, V Ananthaswamy et al [26], discussed the analytical expressions of velocity and heat transfer for a boundary layer flow with thermal radiation past a moving vertical porous plate. The governing non-linear differential equations are solved analytically using Homotopy analysis method

In 2020, Aysegul Ozgur et al. [28] proposed a numerical method for solving elastic stress problems in hollow discs made of functionally graded materials (FGM) that are exposed to linearly increasing temperature dispersion. The FGM disk's modulus of elasticity and coefficient of thermal expansion are assumed to change in various ways along the radial direction.

In 2021, Durmuş Yarımpabuç et al [29] proposed a closed-form solution for transient thermal stress analysis of functionally graded hollow cylinder exposed to high-temperature difference under the influence of periodic rotation.

3. Objectives of proposed research:

1) To study basic concepts of heat transfer and thermal Stress analysis

2) To study some methods to solve boundary value problems for thermal stress analysis

3) To study the analytical and numerical methods.

4) To compare the solution of the boundary value problems by analytical and numerical methods.

5) To study recent applications of boundary value problems.

6) Hence, in the present work, we will try to develop the new analytical results and new iterative techniques for boundary value problems to obtain their solution. These will represent graphically by Mathematica, MATLAB, Python, Scilab and Mathcad.

4. Research Methodology:

The following steps will be including in the methodology:

1) To refer the relevant reference books on partial differential equations, boundary value problems in thermal engineering.

2) To study Mathematica, MATLAB, Python, Scilab, Mathcad softwares.

3) To study analytical solution of boundary value problems in thermal stresses.

4) To study Numerical methods for obtaining approximate solutions of boundary value problems.

5) To discuss the stability and convergence of Numerical methods.

6) To represent Numerical solutions of boundary value problems graphically by Mathematica, MATLAB and discuss their nature.

7) Comparison of solution to validate numerical solution.

5. Plan of work:

Sr. No.	Period	Plan of work	Targets to be Achieved
1	First year	Study of background information relevant to the research issue	Excellent understanding of the subject and well-organized coursework.
2	Second year	Collecting and analysing research articles.	Attempt to obtain new results based on existing work and submit some papers.
3	Third year	Publishing some papers and writing of the thesis.	Some more papers and thesis submission.

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